Problem 1.33

If you have some experience in electromagnetism and with vector calculus, prove that the magnetic forces, $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$, between two steady current loops obey Newton's third law. [*Hints:* Let the two currents be I_1 and I_2 and let typical points on the two loops be $\mathbf{r}_1$ and $\mathbf{r}_2$. If $d\mathbf{r}_1$ and $d\mathbf{r}_2$ are short segments of the loops, then according to the Biot–Savart law, the force on $d\mathbf{r}_1$ due to $\mathbf{r}_2$ is

$$\frac{\mu_o}{4\pi}\frac{I_1 I_2}{s^2}\, d\mathbf{r}_1 \times (d\mathbf{r}_2 \times \hat{\mathbf{s}})$$

where $\mathbf{s} = \mathbf{r}_1 - \mathbf{r}_2$. The force $\mathbf{F}_{12}$ is found by integrating this around both loops. You will need to use the "$BAC - CAB$" rule to simplify the triple product.]

Solution

Get the magnetic force on loop 1 due to loop 2 by integrating the given expression around both loops.

$$\begin{aligned}
\mathbf{F}_{12} &= \oint_{\text{loop 1}} \oint_{\text{loop 2}} \frac{\mu_o}{4\pi}\frac{I_1 I_2}{s^2}\, d\mathbf{r}_1 \times (d\mathbf{r}_2 \times \hat{\mathbf{s}}) \\
&= \frac{\mu_o}{4\pi} I_1 I_2 \oint_{\text{loop 1}} \oint_{\text{loop 2}} \frac{d\mathbf{r}_1 \times (d\mathbf{r}_2 \times \hat{\mathbf{s}})}{s^2}
\end{aligned}$$

Use the BAC-CAB identity: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$.

$$\begin{aligned}
\mathbf{F}_{12} &= \frac{\mu_o}{4\pi} I_1 I_2 \oint_{\text{loop 1}} \oint_{\text{loop 2}} \frac{d\mathbf{r}_2 (d\mathbf{r}_1 \cdot \hat{\mathbf{s}}) - \hat{\mathbf{s}}(d\mathbf{r}_1 \cdot d\mathbf{r}_2)}{s^2} \\
&= \frac{\mu_o}{4\pi} I_1 I_2 \oint_{\text{loop 1}} \oint_{\text{loop 2}} \left[d\mathbf{r}_2 \left(d\mathbf{r}_1 \cdot \frac{\hat{\mathbf{s}}}{s^2} \right) - \frac{\hat{\mathbf{s}}}{s^2}(d\mathbf{r}_1 \cdot d\mathbf{r}_2) \right] \\
&= \frac{\mu_o}{4\pi} I_1 I_2 \left[\oint_{\text{loop 1}} \oint_{\text{loop 2}} d\mathbf{r}_2 \left(d\mathbf{r}_1 \cdot \frac{\hat{\mathbf{s}}}{s^2} \right) - \oint_{\text{loop 1}} \oint_{\text{loop 2}} \frac{\hat{\mathbf{s}}}{s^2}(d\mathbf{r}_1 \cdot d\mathbf{r}_2) \right] \\
&= \frac{\mu_o}{4\pi} I_1 I_2 \left[\oint_{\text{loop 1}} \oint_{\text{loop 2}} d\mathbf{r}_2 \left(\frac{\hat{\mathbf{s}}}{s^2} \cdot d\mathbf{r}_1 \right) - \oint_{\text{loop 1}} \oint_{\text{loop 2}} \frac{1}{s^2}\left(\frac{\mathbf{s}}{s} \right)(d\mathbf{r}_1 \cdot d\mathbf{r}_2) \right] \\
&= \frac{\mu_o}{4\pi} I_1 I_2 \left\{ \oint_{\text{loop 2}} \left[\oint_{\text{loop 1}} \left(\frac{\hat{\mathbf{s}}}{s^2} \cdot d\mathbf{r}_1 \right) \right] d\mathbf{r}_2 - \oint_{\text{loop 1}} \oint_{\text{loop 2}} \frac{\mathbf{s}}{s^3}(d\mathbf{r}_1 \cdot d\mathbf{r}_2) \right\} \\
&= \frac{\mu_o}{4\pi} I_1 I_2 \left\{ \oint_{\text{loop 2}} \underbrace{\left[\oint_{\text{loop 1}} \nabla \left(-\frac{1}{s} \right) \cdot d\mathbf{r}_1 \right]}_{=0} d\mathbf{r}_2 - \oint_{\text{loop 1}} \oint_{\text{loop 2}} \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(d\mathbf{r}_1 \cdot d\mathbf{r}_2) \right\}
\end{aligned}$$

The integral in square brackets is zero because of the fundamental theorem for gradients.

$$\int_{\mathbf{a}}^{\mathbf{b}} \nabla f \, d\tau = f(\mathbf{b}) - f(\mathbf{a})$$

For a closed loop, $\mathbf{b} = \mathbf{a}$, which makes the right side zero.

$$\mathbf{F}_{12} = -\frac{\mu_o}{4\pi} I_1 I_2 \oint_{\text{loop 1}} \oint_{\text{loop 2}} \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} (d\mathbf{r}_1 \cdot d\mathbf{r}_2)$$

Therefore,

$$\begin{aligned}
\mathbf{F}_{21} &= -\frac{\mu_o}{4\pi} I_2 I_1 \oint_{\text{loop 2}} \oint_{\text{loop 1}} \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} (d\mathbf{r}_2 \cdot d\mathbf{r}_1) \\
&= -\frac{\mu_o}{4\pi} I_1 I_2 \oint_{\text{loop 1}} \oint_{\text{loop 2}} \frac{-(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3} (d\mathbf{r}_1 \cdot d\mathbf{r}_2) \\
&= \frac{\mu_o}{4\pi} I_1 I_2 \oint_{\text{loop 1}} \oint_{\text{loop 2}} \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} (d\mathbf{r}_1 \cdot d\mathbf{r}_2) \\
&= -\mathbf{F}_{12}.
\end{aligned}$$

The loops are separate and have their own parameterizations, so the integrals over them can be interchanged without issue.